"Special Theory of Relativity" – A Formulae Guide

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(Dated: October 10, 2017)

Here, I summarize all formulae with some basics of Special Theory of Relativity. For detailed calculations, see reference sections.

I. INTRODUCTION

Consider, a frame S' is moving w.r.t. a rest frame S with a constant velocity v along x-direction (For simplicity, we first take boost along x-axis, later we take general boost direction).

Einstein's Postulates:

- Laws of Physics is frame-independent so physics should be same in all inertial frame.
- Speed of light (c) is invariant in all inertial frame.

Now, we introduce two quantity for writing my all formulae in a simple and symmetric form:

$$\beta = \frac{v}{c}$$

and

$$\gamma = (1 - \beta^2)^{-\frac{1}{2}}$$

A point in 4-dimensional manifold is represented by in S, S'

$$x^\mu \equiv (x^0, x^i)$$

$$x'^\mu \equiv (x'^0, x'^i)$$

where μ=0,1,2,3 and i=1,2,3

- Here, we took **Bjorken-Drell metric with signature = (+,-,-,-)** s.t

$$x_\mu \equiv (x_0, -x_i)$$

and

$$x_\nu = \eta_{\nu\mu} x^\mu$$

$$x^\nu = \eta^{\nu\mu} x_\mu$$

- x^0 is time component and represented by $x^0 = ct$ in Minkwoski's Space-time diagram.

II. LORENTZ TRANSFORMATION

II.1. Boost along x-direction

If we apply boost along x-direction, space along perpendicular to the boost remain unaffected by time.

$$\boxed{\Delta x'^0 = \gamma(\Delta x^0 - \beta \Delta x)}$$

$$\boxed{\Delta x' = \gamma(\Delta x - \beta \Delta x^0)}$$

$$\boxed{\Delta y' = \Delta y}$$

$$\boxed{\Delta z' = \Delta z}$$

In matrix form,

$$\begin{bmatrix} \Delta x'^0 \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x^0 \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}$$

In other notation,

$$\Delta x'^\mu = \Lambda^\mu_\nu \Delta x^\nu$$

- **We can invert this above relation to express coordinates of S in terms of S' by introducing -ve boost means replacing $\vec{\beta}$ by -$\vec{\beta}$.**
- Introducing **Rapidity** (ρ), we can rewrite the above eqns

$$\begin{bmatrix} x^0(\rho) \\ x(\rho) \end{bmatrix} = \begin{bmatrix} Cosh\rho & Sinh\rho \\ Sinh\rho & Cosh\rho \end{bmatrix} \begin{bmatrix} x^0(\rho = 0) \\ x(\rho = 0) \end{bmatrix}$$

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where ρ is related to the boost by

$$\boxed{tanh(\rho) = -\beta}$$

with

$$Sinh(\rho) = -\beta\gamma$$

and

$$Cosh(\rho) = \gamma$$

- So, we can see rapidity is a measure of boost in hyperbolic geometry.
- This matrix looks like rotation matrix but it's not as hyperbolic function isn't periodic function. That's why elements of this matrix don't form a compact group. Hence,it is called **Pseudo-rotation**.

II.2. General Lorentz Boost

If Lorentz boost is applied in any arbitrary direction,we can always decompose the motion of a body along the boosted direction means parallel to $\vec{\beta}$ and perpendicular to $\vec{\beta}$.

$$\Delta\vec{r} = \Delta\vec{r}_{||} + \Delta\vec{r}_{\perp}$$

s.t.

$$\Delta\vec{r}_{\perp}.\hat{\beta} = 0$$

$$\Delta\vec{r}_{||} = (\Delta\vec{r}.\vec{\beta})\hat{\beta} = |\Delta\vec{r}_{||}|\hat{\beta}$$

Our Lotrentz equations become

$$\boxed{\Delta x'^0 = \gamma(\Delta x^0 - |\vec{\beta}|.|\Delta\vec{r}_{||}|)}$$

$$\boxed{\Delta r'_{||} = \gamma(\Delta\vec{r}_{||} - |\vec{\beta}|.\Delta x^0)}$$

$$\boxed{\Delta\vec{r'}_{\perp} = \Delta\vec{r}_{\perp}}$$

Last equation is vector equation as it contains two equations along two perpendicular direction of $\vec{\beta}$.

In matrix form,

$$\begin{bmatrix} \Delta x'^0 \\ \Delta r'_{||} \\ \Delta\vec{r'}_{\perp} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma|\vec{\beta}| & 0 \\ -\gamma|\vec{\beta}| & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x^0 \\ \Delta r_{||} \\ \Delta\vec{r}_{\perp} \end{bmatrix}$$

III. MINKWOSKI DIAGRAM

In Bjorken-Drell metric notation, signature is (+,-,-,-) and sapce-time interval Δs^2 is invariant.

$$\Delta s^2 = (\Delta x^0)^2 - (\Delta\vec{r})^2 = (\Delta x^{0\prime})^2 - (\Delta\vec{r'})^2 = \Delta s'^2$$

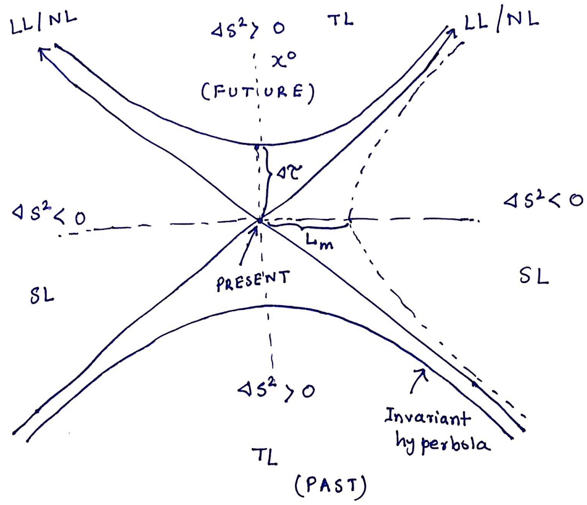

FIG. 1. Minkwoski Diagram

- Time-like (TL)

$$\Delta s^2 > 0$$

$x^0 > 0 \rightarrow$ Future;
$x^0 = 0 \rightarrow$ Present;
$x^0 < 0 \rightarrow$ Past.

- Null-like (NL)/Light-like (LL)

$$\Delta s^2 = 0$$

- Space-like (SL)

$$\Delta s^2 < 0$$

III.1. Length Contraction

Length of a body is **minimum** (L_m) in that moving frame from which one can measure two ends of the body **simultaneously**. Any length $(L^{(\gamma)})$ measured from other Lorentz frame gives will be more than L_m. This L_m is called **Proper length**.

$$\boxed{L_m = \frac{L^{(\gamma)}}{\gamma}}$$

III.2. Time Dilation

Time between two events is **minimum** (τ) **in rest frame** (i.e. two events are happening in same position). From any other Lorentz frame, this time ($t^{(\gamma)}$) will be dialated. τ is called **Proper time**.

$$\boxed{t^{(\gamma)} = \gamma.\tau}$$

IV. GENERAL VELOCITY TRANSFORMATION

A particle having velocity **u** (for particle, $\vec{\beta}_u$) has been observed in S and $\tilde{S}$ where $\vec{\beta}$ be the relative velocity between two frames.

Einstein's velocity addition rule says **parallel component** should transforms like,

$$\boxed{\vec{\beta}'_u = \frac{[(\vec{\beta}_u.\hat{\beta})\hat{\beta} - \vec{\beta}]}{[1 - \vec{\beta}_u.\vec{\beta}]}}$$

- If $\vec{\beta}_u$ is parallel to $\vec{\beta}$, then

$$\boxed{\vec{\beta}'_u = \hat{\beta}\frac{\beta_u - \beta}{1 - \beta_u\beta}}$$

item If $\vec{\beta}_u$ is anti-parallel to $\vec{\beta}$,then put $-\vec{\beta}$ instead of $\vec{\beta}$.

- In Non-relativistic limit $u, v << c$, the above formula becomes the Gallilean relative velocity

$$u' = u - v$$

- Also, particle's

$$\gamma'_u = \gamma\gamma_u[1 - \vec{\beta}_u.\vec{\beta}]$$

whether **perpendicular component** should transform like

$$\boxed{\vec{\beta}'_{u_\perp} = \frac{\vec{\beta}_{u_\perp}}{\gamma(1 - \vec{\beta}_u.\vec{\beta})}}$$

V. FOUR-VECTOR

- $\breve{A}$ be a 4-vector: $\breve{A} \equiv (A^0, \vec{A})$
- $\breve{A}.\breve{B} = \eta_{\mu\nu}A^\mu B^\nu = \eta^{\mu\nu}A_\mu B_\nu$ $= A^0B^0 - \vec{A}.\vec{B}$
- **Four-velocity:**

$$\boxed{\breve{u}^\alpha = \frac{d\breve{x}^\alpha}{d\tau} = \gamma_u\breve{v}^\alpha}$$

So, $\breve{u}^\alpha \equiv (u^0, \vec{u}) = (\gamma_u c, \gamma_u c\vec{\beta}_u)$

- $\breve{u}.\breve{u} = c^2 \rightarrow$TL 4-vector.
- **Four-acceleration:**

$$\boxed{\breve{a}^\alpha = \frac{d\breve{u}^\alpha}{d\tau}}$$

- $$\breve{u}.\breve{a} = \breve{u}.\frac{d\breve{u}}{d\tau} = \frac{1}{2}\frac{d}{d\tau}(\breve{u}.\breve{u}) = 0 \Rightarrow \boxed{\breve{a} \perp \breve{u}}$$

- $$\boxed{\frac{d\gamma_u}{d\tau} = \gamma_u^3(\vec{\beta}_u.\dot{\vec{\beta}}_u)}$$

where $\dot{\vec{\beta}}_u = \frac{d\vec{\beta}_u}{dt}$

- $$\boxed{\breve{a} \equiv (a^0, \vec{a}) = [\gamma c\gamma_u^3(\vec{\beta}_u.\dot{\vec{\beta}}_u), \gamma c\frac{d}{dt}(\gamma_u\vec{\beta}_u)]}$$

- $\breve{a}.\breve{a} < 0 \rightarrow$ SL 4-vector.
- **Four-momentum:**

$$\boxed{\breve{p}^\mu = m\breve{u}^\mu \equiv (\frac{E}{c}, \vec{p}) = (\gamma_u mc, \gamma_u mc\vec{\beta}_u)}$$

- Energygy $\boxed{E = \gamma_u mc^2}$
- $$p^2 \equiv \breve{p}.\breve{p} = m^2|\breve{u}|^2 = m^2c^2$$

$$\Rightarrow |\breve{p}|^2 = p^\mu p_\mu = m^2c^2 \rightarrow \text{TL 4-vector.}$$

- **Four-force:**

$$\boxed{\breve{f}^\mu = m\breve{a}^\mu = \frac{d\breve{p}^\mu}{d\tau} \equiv (\gamma\vec{F}.\vec{\beta}_u, \gamma\vec{F})}$$

  where $\vec{\beta}_u, \vec{F}$ are three-velocity,three-force.

- **Four-force and Four-acceleration are not parallel in general**.

$$\boxed{\vec{a} = \frac{\vec{F} - (\vec{F}.\vec{\beta}_u)\vec{\beta}_u}{m\gamma}}$$

VI. MASS

We can redefine mass of the particle (older concept):

- **Relativistic mass:** $\boxed{m^{(\gamma)} = \gamma.m_0}$
- **Longitudinal mass:** From the above relation between relativistic force and acceleration,if $\vec{F}||\vec{\beta}_u$, then we can introduce that $\vec{F}_{||} = m_{||}a_{||}$ where **Longitudinal mass** is

$$\boxed{m_{||} = \gamma^3 m}$$

- **Transverse mass:** If $\vec{F} \perp \vec{\beta}_u$, then we can introduce that $\vec{F}_\perp = m_\perp a_\perp$ where **Transverse mass** is

$$\boxed{m_\perp = \gamma m}$$

VII. ENERGY-MOMENTUM RELATION

- Total energy of a massive particle:

$$\boxed{E = \sqrt{\vec{p}^2c^2 + m^2c^4}}$$

  with rest energy $E_0 = mc^2$

- $$E = mc^2(1 + \frac{p^2}{m^2c^2})^{1/2} = mc^2 + K.E$$

- If rest energy >> K.E. then $E \approx mc^2 + \frac{p^2}{2m} \Rightarrow$ NR expression
- For massless particle like photon (m=0), $\boxed{E = pc}$

VIII. LAGRANGIAN OF FREE PARTICLE

Lagrangian of a free particle can be written as

$$\boxed{\mathcal{L} = -mc^2\sqrt{1-\beta^2}}$$

and action is

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}dt$$

IX. TRANSFORMATION OF (E,P) → (E′, P′)

We can also represent energy and 3-momentum of S' in terms of S.It's simply lorentz transformation of four-momentum.

$$\begin{bmatrix} \Delta p'^0 = \Delta E' \\ \Delta p'_{||} \\ \Delta \vec{p'}_\perp \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma|\vec{\beta}| & 0 \\ -\gamma|\vec{\beta}| & \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta p^0 = \Delta E \\ \Delta p_{||} \\ \Delta \vec{p}_\perp \end{bmatrix}$$

X. TRANSFORMATION OF ACCELERATION AND FORCE FOR GENERAL BOOST

X.1. Acceleration:

Parallel component

$$\boxed{a'_{||} = \frac{a_{||}}{\gamma^3(1 - \vec{\beta}.\vec{\beta}_u)^3}}$$

Perpendicular component

$$\boxed{\vec{a}'_\perp = \frac{a_\perp}{\gamma^2(1 - \vec{\beta}.\vec{\beta}_u)^2} + \frac{\vec{\beta}_{u_\perp}(\vec{\beta}.\vec{a}_{||})}{\gamma^2(1 - \vec{\beta}.\vec{\beta}_u)^3}}$$

X.2. Force:

Here,

$$\vec{F} = \vec{F}_{||} + \vec{F}_\perp$$

Parallel component

$$\boxed{F'_{||} = \frac{F_{||} - (\vec{F}.\vec{\beta}_u)|\vec{\beta}|}{(1 - \vec{\beta}.\vec{\beta}_u)}}$$

Perpendicular component

$$\boxed{\vec{F}'_\perp = \frac{F_\perp}{\gamma(1-\vec{\beta}.\vec{\beta}_u)}}$$

XI. DOPPLER EFFECT

If source and observer moving in same direction,then

$$\nu = \nu' \frac{\sqrt{1-\beta^2}}{1-\beta cos\theta}$$

- Source and observer moves towards each other, $\theta = 0^o$

$$\nu = \nu' \sqrt{\frac{1+\beta}{1-\beta}}$$

- Source and observer moving away from each other, $\theta = 180^o$,

$$\nu = \nu' \sqrt{\frac{1-\beta}{1+\beta}}$$

- If source and observer moving perpendicularly to each other, $\theta = 90^o$,

$$\nu = \nu' \sqrt{1-\beta^2}$$

XII. ABERRATION

Relativistic aberration is

$$tan\theta' = \frac{sin\theta\sqrt{1-\beta^2}}{cos\theta - \beta}$$

Inverse aberration is

$$tan\theta = \frac{sin\theta'\sqrt{1-\beta^2}}{cos\theta' + \beta}$$

XIII. RELATIVISTIC EM FIELD

XIII.1. Transformation of E-B Fields

Lorentz transformation of EM field follows same transformation as any vector. Let's assume propagation is field is in y-z plane,so that our E-B field can be decomposed in E_y, E_z, B_y, B_z. The perpendicular fields remain unchanged.

$$E'_x = E_x; B'_x = B_x$$

The parallel fields change.

$$E'_y = \gamma(E_y - \beta c B_z)$$
$$E'_z = \gamma(E_z + \beta c B_y)$$
$$cB'_y = \gamma(cB_y + \beta E_z)$$
$$cB'_z = \gamma(cB_z - \beta E_y)$$

XIII.2. Current-Density J^μ

Due to Lorentz boost,charge density (ρ) and current density ($\vec{J} = \rho c\vec{\beta_u}$) changes in a moving frame having velocity $\vec{\beta}$. We already know 4-velocity $\breve{u} \equiv (\gamma c, \gamma c\beta_u)$

Now,charge density and current density changes like

$$\rho' = \rho\gamma; \vec{J}' = \gamma c\beta_u\rho$$

So,we can construct a **current-density 4-vector**

$$\boxed{\breve{J} = \rho\breve{u} \equiv (c\rho, \vec{J})}$$

XIII.3. Continuity Equation

We know continuity equation is

$$\vec{\nabla}.\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

We can rewrite this

$$\Sigma_i \frac{\partial \vec{J}^i}{\partial x^i} + \frac{1}{c}\frac{\partial J^0}{\partial t} = 0$$

$$\Rightarrow \boxed{\frac{\partial J^\mu}{\partial x^\mu} = 0}$$

- It is 4-dimensional divergence. So **Continuity equation says that current density 4- vector is divergenceless.**

XIII.4. Field Tensor

- We need an **anti-symmetric,second rank tensor** to represent E-B field.

From the above transformation, we can construct two **field tensors** $F^{\mu\nu}, G^{\mu\nu}$. $G^{\mu\nu}$ is **dual-tensor** of $F^{\mu\nu}$.

$$F^{\mu\nu} = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix}$$

$$G^{\mu\nu} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{bmatrix}$$

XIII.5. Maxwell's Equation

Four Maxwell's Equations in Electrodynamics can be written in compact form using this field tensors and current density.

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu}; \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0$$

- The first equation gives us Gauss's law and Ampere's law: $\vec{\nabla}.\vec{E} = \rho/\epsilon_0$ and $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ for all μ.
- While the second one gives us rest two equations: $\vec{\nabla}.\vec{B} = 0$ and $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

XIII.6. Relativistic Potential A^{μ}

We know E and B can be written in terms of Potentials.

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

and

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

We can reformulate these potentials in compact form

$$A^{\mu} \equiv (\frac{\phi}{c}, A_i)$$

Transformation of A^{μ} is

$$\frac{\phi'}{c} = \gamma(\frac{\phi}{c} - \beta A_x)$$

$$A'_x = \gamma(A_x - \beta\frac{\phi}{c})$$

$$A'_{y,z} = A_{y,z}$$

XIV. REFERENCE

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